

# A Continuous Inventory Problem

HERBERT HAUPTMAN AND ARTHUR ZIFFER

*Mathematical Physics Branch  
Mathematics and Information Sciences Division*

February 21, 1968



**NAVAL RESEARCH LABORATORY**  
**Washington, D.C.**



## REFERENCE

- [1] Denicoff, M., Fennell, J., Haber, S. E., Marlow, W. H., Segel, F. W., and Solomon, Henry, "A Polaris Logistics Model," Naval Research Logistics Quarterly, Vol. 11, No. 4, December, 1964.

constitute a solution of the system (9) and (10). This solution is obviously unique since any other assumed solution would determine by (22) a different value for  $k$  satisfying also (28) and (29), a contradiction.

We conclude by applying our general method to the linear distribution functions (11) considered earlier. Using the middle inequalities of (11) in (23) we can solve (23) explicitly for  $s_i = s_i(k)$ :

$$(30) \quad s_i = \frac{B_i - E_i a_i}{E_i b_i} + \frac{v_i k}{E_i b_i}.$$

Using equations (27) and (29), we obtain

$$(31) \quad V = \sum_{i=1}^n v_i \left( \frac{B_i - E_i a_i}{E_i b_i} + \frac{v_i}{E_i b_i} k \right),$$

which can be solved for  $k$ :

$$(32) \quad k = \frac{V - \sum_{i=1}^n v_i \left( \frac{B_i - E_i a_i}{E_i b_i} \right)}{\sum_{i=1}^n \frac{v_i^2}{E_i b_i}} = \frac{V - V^*}{\sum_{i=1}^n \frac{v_i^2}{E_i b_i}}$$

Substituting from (32) into (30) we get (20), obtained earlier by a more laborious computation.

#### Acknowledgement

Thanks are due to Phillip W. Mast and Burton N. Navid for their helpful comments.

range of  $F_i(s_i)$ , the interval  $[0,1]$ , is also the domain of  $F_i^{-1}$ . Hence the domain of  $s_i(k)$  is determined by

$$(25) \quad 0 \leq \frac{kv_i + B_i}{A_i + B_i} \leq 1$$

and is therefore the interval

$$(26) \quad -\frac{B_i}{v_i} \leq k \leq \frac{A_i}{v_i} .$$

Thus, as  $k$  ranges from  $-\frac{B_i}{v_i}$  to  $\frac{A_i}{v_i}$ ,  $s_i(k)$  is continuous and strictly increasing from  $-\infty$  to  $+\infty$ . Therefore the function

$$(27) \quad V(k) = \sum_{i=1}^n v_i s_i(k) ,$$

defined over the domain

$$(28) \quad a = \max_i \left\{ -\frac{B_i}{v_i} \right\} \leq k \leq \min_i \left\{ \frac{A_i}{v_i} \right\} = b ,$$

is continuous and strictly increasing from  $-\infty$  to  $+\infty$ . (We note that since  $A_i > 0$  and  $B_i > 0$ ,  $a < 0$  and  $b > 0$ ; hence,  $[a,b]$  is non-vacuous.) Therefore the equation

$$(29) \quad V(k) = V ,$$

has a unique solution for  $k$  satisfying (28). Clearly this value of  $k$  determines unique values for the  $s_i$ , by means of (24), which evidently

The solution (20) of (12) and (9) makes plausible the following theorem, the main result of this paper:

**Theorem.** If each  $F_i(s_i)$  is continuous and strictly increasing, then the system of equations (9) and (10) have a unique solution  $s_1, s_2, \dots, s_n$ .

**Proof:** Assume first that there exist a set of  $s_i$ 's satisfying (9) and (10). The right-hand side of (10) is independent of  $i$ ; let us, therefore, call it  $k$ . Then (10) becomes

$$(22) \quad \frac{(A_i+B_i)F_i(s_i) - B_i}{v_i} = k,$$

so that

$$(23) \quad F_i(s_i) = \frac{k v_i + B_i}{A_i + B_i}.$$

Now the right-hand side of (23) is continuous and strictly increasing as a function of  $k$ . Since we are assuming that the  $F_i(s_i)$  are also continuous and strictly increasing, the inverse functions  $F_i^{-1}$  exist and are continuous and strictly increasing also. Hence each  $s_i(k)$ , defined by:

$$(24) \quad s_i = s_i(k) = F_i^{-1} \left( \frac{k v_i + B_i}{A_i + B_i} \right),$$

being the composition of two functions, each continuous and strictly increasing, is itself continuous and strictly increasing. Since the domain of each  $F_i(s_i)$  is  $(-\infty, +\infty)$ , the range of  $F_i^{-1}$  and therefore of  $s_i(k)$  is also  $(-\infty, +\infty)$ . The

From (15) it is seen that the root (16),  $S = 0$ , leads to

$$(18) \quad s_i^* = \frac{B_i - E_i a_i}{E_i b_i},$$

which is what we would have obtained if we had used (11) (middle inequalities) in (5):

$$(19) \quad a_i + b_i s_i^* = \frac{B_i}{A_i + B_i} = \frac{B_i}{E_i},$$

that is to say, (18) is the "unconstrained" solution of (12). Since the solution (18) of (12) does not satisfy the constraint condition (9), we turn to the value of  $S$  given by (17) which, using (15), leads to (in view of (6))

$$(20) \quad s_i = s_i^* - \frac{v_i(V^* - V)}{E_i b_i \sum_{j=1}^n \frac{v_j^2}{E_j b_j}}, \quad i = 1, 2, \dots, n.$$

Not only is it now easily verified that the solution (20) of (12) satisfies also the constraint condition (9), but, in view of  $V < V^*$ , we find that  $s_i < s_i^*$ , a result that might have been anticipated. Furthermore, (20) shows that

$$(21) \quad \lim_{V \rightarrow V^*} s_i = s_i^*.$$

Under the linearization (11), equations (10) become the system of quadratics in the  $s_i$ ,

$$(12) \quad \frac{1}{V} \left\{ \sum_{j=1}^n E_j b_j s_j^2 + \sum_{j=1}^n (E_j a_j - B_j) s_j \right\} - \frac{E_i b_i}{v_i} s_i - \frac{E_i a_i - B_i}{v_i} = 0, \\ 1 = 1, 2, \dots, n,$$

in which we have replaced  $A_i + B_i$  by  $E_i$ . Writing

$$(13) \quad S = \sum_{j=1}^n E_j b_j s_j^2 + \sum_{j=1}^n (E_j a_j - B_j) s_j,$$

(12) becomes

$$(14) \quad \frac{S}{V} - \frac{E_i b_i}{v_i} s_i - \frac{E_i a_i - B_i}{v_i} = 0.$$

Solving (19) for  $s_i$  gives

$$(15) \quad s_i = \frac{S v_i}{V E_i b_i} + \frac{B_i - E_i a_i}{E_i b_i}.$$

Substituting for the  $s_i$  from (15) into (13) yields a quadratic equation in  $S$  having the two roots:

$$(16) \quad S = 0,$$

$$(17) \quad S = \frac{V \left( V - \sum_{i=1}^n v_i \frac{B_i - E_i a_i}{E_i b_i} \right)}{\sum_{i=1}^n \frac{v_i^2}{E_i b_i}}.$$

We note that (5) satisfies the system (10). However in view of  $V < V^*$ , this solution does not satisfy the equation of constraint, (9). We therefore anticipate that (10) has two solutions, one of which satisfies (9).

Before investigating the system (10) in full generality, it is instructive to consider first a suitable linearization which not only permits explicit solutions for the  $s_i$  to be obtained, but also clarifies the relationship between the two solutions of (10) that are anticipated. The appropriate linearization is obtained by taking the  $f_i(x)$  to be rectangular distributions so that the  $F_i(s_i)$  are of the form

$$(11) \quad F_i(s_i) = \begin{cases} 0 & \text{if } s_i \leq -\frac{a_i}{b_i} , \\ a_i + b_i s_i & \text{if } -\frac{a_i}{b_i} \leq s_i \leq \frac{1-a_i}{b_i} , \\ 1 & \text{if } \frac{1-a_i}{b_i} \leq s_i , \end{cases}$$

where the  $a_i$  and  $b_i$  are suitable constants. This linearization is particularly appropriate in the case that the given probability distributions have the property that they are unimodal, whence their distribution functions have only one inflection point, in the neighborhood of which the distribution functions are approximately linear. It is also assumed that the desired solution (the one that satisfies the constraint (9)) satisfies the middle inequalities of (11), a plausible assumption provided that  $V$  is not too small. (The precise condition is readily obtained from (20), but is not important for our present purpose.)

and assuming that for each  $s_i$  there is no constraint condition, it is readily verified that  $\phi_i(s_i)$  attains its minimum at the  $s_i^*$  satisfying

$$(5) \quad F_i(s_i^*) = \frac{B_i}{A_i + B_i}, \quad i = 1, 2, \dots, n.$$

If  $V \geq V^*$  for  $V^*$  defined by

$$(6) \quad V^* = \sum_{i=1}^n v_i s_i^*$$

and the  $s_i^*$  defined by (5), then  $\phi$  attains its minimum at  $(s_1^*, s_2^*, \dots, s_n^*)$  since  $\phi$  is separable and

$$(7) \quad \min_{\substack{s_1, s_2, \dots, s_n \\ \sum v_i s_i = V}} \phi(s_1, s_2, \dots, s_n) = \sum_{i=1}^n \min_{s_i} \phi_i(s_i).$$

In the case of  $V < V^*$  we use the Lagrange multiplier ( $\lambda$ ) technique and obtain the minimizing  $s_i$  as those that satisfy the following system of equations

$$(8) \quad A_i F_i(s_i) + B_i [F_i(s_i) - 1] + \lambda v_i = 0, \quad i = 1, 2, \dots, n$$

$$(9) \quad \sum_{i=1}^n v_i s_i = V$$

which, after eliminating  $\lambda$ , become

$$(10) \quad \frac{(A_i + B_i) F_i(s_i) - B_i}{v_i} = \frac{\sum_{j=1}^n s_j \left\{ (A_j + B_j) F_j(s_j) - B_j \right\}}{V}, \quad i = 1, 2, \dots, n.$$

For positive  $A_i$ ,  $B_i$ , non-negative  $v_i$ ,  $s_i$ , continuous probability densities  $f_i(x)$  such that

$$(1) \quad \int_{-\infty}^{\infty} x f_i(x) dx < \infty$$

and

$$(2) \quad \phi_i(s_i) = A_i \int_{-\infty}^{s_i} (s_i - x) f_i(x) dx + B_i \int_{s_i}^{\infty} (x - s_i) f_i(x) dx,$$

$$i = 1, 2, \dots, n,$$

we seek to minimize

$$(3) \quad \phi(s_1, s_2, \dots, s_n) = \sum_{i=1}^n \phi_i(s_i)$$

subject to the constraint

$$\sum_{i=1}^n v_i s_i = V.$$

Letting

$$(4) \quad F(s) = \int_{-\infty}^{s_i} f_i(x) dx$$

# A Continuous Inventory Problem

by

Herbert Hauptman and Arthur Ziffer

A submarine on patrol must stock various amounts of  $n$  different supply items or replacement parts. If  $s_i$  is the amount of item  $i$  stocked and  $f_i(j)$  is the probability that precisely  $j$  units of item  $i$  will be demanded on the patrol then

$$\sum_{j=0}^{s_i} (s_i - j)f_i(j) \quad \text{and} \quad \sum_{j=s_i+1}^{s_i} (j - s_i)f_i(j)$$

are the expected amounts of overstocking and understocking of item  $i$ , respectively, that will occur. If  $A_i$  and  $B_i$  are the unit costs of overstocking and understocking item  $i$ , respectively, then the "cost" function for item  $i$  is [1]

$$\phi(s_i) = A_i \sum_{j=0}^{s_i} (s_i - j)f_i(j) + B_i \sum_{j=s_i+1}^{\infty} (j - s_i)f_i(j)$$

and the "cost" function for the stock vector  $(s_1, s_2, \dots, s_n)$  is

$$\phi(s_1, s_2, \dots, s_n) = \sum_{i=1}^n \phi_i(s_i) .$$

Finally, introducing a total volume constraint  $V$ , with  $v_i$  the volume of one unit of item  $i$ , and casting the whole situation into continuous form we obtain the problem to be discussed in this report.

## ABSTRACT

A basic inventory model which is concerned with stocking various amounts of  $n$  different items for a submarine going on patrol is considered. There are costs for overstocking and understocking each item, a probability distribution which specifies the probability that any number of each item will be required, and an overall volume constraint. The discrete model is then cast into continuous form and the resulting problem in constrained minimization is solved.

## PROBLEM STATUS

This is a final report on one phase of the problem; work on this problem is continuing.

## AUTHORIZATION

NRL Problem B01-03

Project RR 002-10-45-5060

## CONTENTS

ABSTRACT .....	ii
PROBLEM STATUS .....	ii
AUTHORIZATION .....	ii
A CONTINUOUS INVENTORY PROBLEM .....	1
REFERENCE .....	10

## DOCUMENT CONTROL DATA - R &amp; D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

Naval Research Laboratory  
Washington, D.C. 20390

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

3. REPORT TITLE

A CONTINUOUS INVENTORY PROBLEM

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Final report on one phase of the problem; work continues on the problem.

5. AUTHOR(S) (First name, middle initial, last name)

Herbert Hauptman and Arthur Ziffer

6. REPORT DATE

February 21, 1968

7a. TOTAL NO. OF PAGES

14

7b. NO. OF REFS

1

8a. CONTRACT OR GRANT NO.

NRL Problem B01-03

b. PROJECT NO.

RR 002-10-45-5060

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

NRL Report 6689

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

This document has been approval for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

Office of Naval Research  
Washington, D.C.

13. ABSTRACT

A basic inventory model which is concerned with stocking various amounts of  $n$  different items for a submarine going on patrol is considered. There are costs for overstocking and understocking each item, a probability distribution which specifies the probability that any number of each item will be required, and an overall volume constraint. The discrete model is then cast into continuous form and the resulting problem in constrained minimization is solved.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Inventory problems Logistics Optimization Submarine inventory problems						